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THREE-DIMENSIONAL QUANTUM MECHANICS IN THE ENERGY DOMAIN: A TIME FREE TREATMENT OF THE HYDROGEN ATOM

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Abstract

In earlier work we proposed a time-free reformulation of quantum mechanics in which the usual temporal evolution is replaced by an auxiliary evolution in the energy domain. The purpose of the present paper is to extend that idea to the full three-dimensional Schrödinger equation and to test it on the hydrogen atom, which is the standard benchmark for any formulation of nonrelativistic quantum mechanics.

We show that the energy-domain wavefunction separates into a stationary spatial eigenfunction and an energy-dependent phase. As a result, the spatial problem remains exactly the same as in the conventional stationary theory. The angular sector still gives the spherical harmonics, the radial Coulomb equation keeps its standard form, and the familiar hydrogenic spectrum, degeneracies, and radial eigenfunctions are recovered without modification.

The analysis also shows that the auxiliary parameter z contributes only a global phase and does not affect probability densities, expectation values, or spectral differences. For time-independent Hamiltonians, the present construction is therefore not a new dynamics but an equivalent representation of standard quantum mechanics written without explicit time. The hydrogen atom and its spectrum provide a direct and complete check of that equivalence.

Keywords: time-free quantum mechanics; energy-domain evolution; three-dimensional Schrödinger equation; hydrogen atom; spherical harmonics; associated Laguerre polynomials; stationary states

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1 Introduction

In standard nonrelativistic quantum mechanics, time appears as the external parameter that drives the evolution of the wavefunction through the Schrödinger equation. That formulation is extremely successful, but the role of time has also been questioned in a number of foundational discussions, especially in settings where one expects time to be emergent rather than fundamental [6, 1, 7, 5, 2]. The motivation of the present work comes from that line of thought. The question we address is simple: for stationary quantum systems, how much of the physical content of the theory really depends on writing the evolution in terms of time?

In our earlier paper we proposed a reformulation in which time is removed and the evolution is written in the energy domain instead [4]. The goal was not to alter the measurable content of quantum mechanics, but to reorganize the formalism so that the usual stationary eigenvalue equation emerges directly from differentiation with respect to energy. In that construction the modified generator is written as

$$\widehat{H}_b = \frac{i\hbar}{2z}\frac{\partial}{\partial E}, \tag{1}$$

and the corresponding wavefunction is taken to be

$$\Psi(\mathbf{r}, E) = \psi(\mathbf{r}) \exp\left(-\frac{iE^2 z}{\hbar}\right), \tag{2}$$

where z is an auxiliary evolution label in energy space.

The form of $\widehat{H}_b$ is fixed by consistency with the phase in (Eq. 2). Indeed,

$$\frac{\partial}{\partial E}\exp\left(-\frac{iE^2 z}{\hbar}\right) = -\frac{2iEz}{\hbar}\exp\left(-\frac{iE^2 z}{\hbar}\right),$$

so the factor $1/(2z)$ ensures that the operator returns the usual energy eigenvalue E. In this way, the dependence on z is confined to a global phase, while the spatial part of the theory remains the same as in ordinary quantum mechanics.

The goal of the present paper is to extend this time-free construction to three dimensions and then apply it to the hydrogen atom. This is the

natural test case. It contains angular momentum, separation in spherical coordinates, a Coulomb potential, associated Laguerre polynomial solutions, and a discrete bound-state spectrum that is known in closed form. If the energy-domain formulation is to be taken seriously, it must reproduce all of these features exactly.

We therefore begin with the three-dimensional energy-based Schrödinger equation

$$\frac{i\hbar}{2z}\frac{\partial \Psi}{\partial E} = \left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\Psi, \tag{3}$$

and substitute the ansatz (Eq. 2). The result is the purely spatial eigenvalue problem

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(\mathbf{r}) = E\,\psi(\mathbf{r}), \tag{4}$$

which is identical to the standard stationary Schrödinger equation [3, 8]. All observables therefore come from $\psi(\mathbf{r})$, while the energy-dependent phase plays no direct physical role.

For the hydrogen atom, the corresponding radial equation becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}\right]u(r) = E\,u(r), \tag{5}$$

with $u(r) = rR(r)$. Solving this equation by the standard asymptotic and power-series method leads to the familiar hydrogenic spectrum

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots, \tag{6}$$

together with the usual associated Laguerre polynomial solutions.

The paper is organized as follows. In Section 2 we derive the three-dimensional form of the energy-domain equation and separate it in spherical coordinates. Section 3 discusses observables and the role of the auxiliary parameter z. Section 4 compares the present construction with the standard time-dependent Schrödinger equation. Section 5 applies the method to the hydrogen atom, and Section 7 shows explicitly how the known hydrogen spectrum is recovered. We then discuss the scope of the formulation and summarize the main conclusions.

2 Energy-Domain Schrödinger Equation in Three Dimensions

We now write the energy-based formulation in full three-dimensional form. The main point of this section is that once the energy-dependent phase is factored out, the remaining spatial problem is exactly the same one that appears in conventional stationary quantum mechanics. For that reason, the whole angular momentum structure survives unchanged.

For a particle of mass m moving in a central potential $V(r)$, the spatial eigenvalue problem is

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V(r)\right]\psi(\mathbf{r}) = E\,\psi(\mathbf{r}), \tag{7}$$

where $\mathbf{r} = (r, \theta, \phi)$ denotes the spherical coordinates shown in Fig. 1.

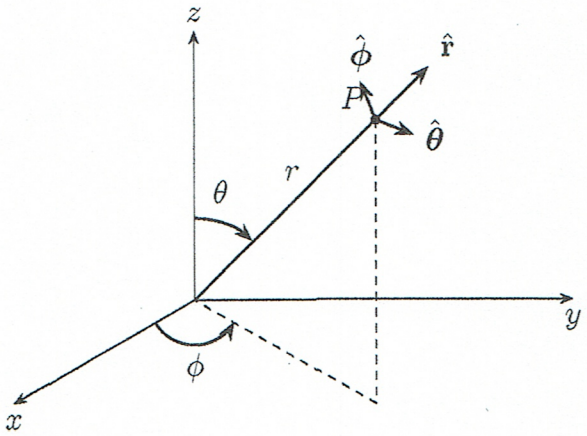

Figure 1: Spherical coordinates showing the radial distance r, polar angle θ, and azimuthal angle ϕ.

The Laplacian in spherical coordinates is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\,\frac{\partial^2}{\partial\phi^2}. \tag{8}$$

Substituting (Eq. 8) into (Eq. 7) gives

$$-\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right]+V(r)\psi=E\psi. \tag{9}$$

To separate the variables we use the standard product ansatz

$$\psi(\mathbf{r})=R(r)\,Y(\theta,\phi), \tag{10}$$

where $R(r)$ contains the radial dependence and $Y(\theta,\phi)$ contains the angular dependence. Inserting (Eq. 10) into (Eq. 9) and dividing by $R(r)Y(\theta,\phi)$ yields

$$-\frac{\hbar^2}{2m}\left[\frac{1}{Rr^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{Yr^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Yr^2\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right]+V(r)=E. \tag{11}$$

After multiplying by $2mr^2/\hbar^2$, the equation separates as

$$-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}\left[V(r)-E\right]=\frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}\right]. \tag{12}$$

The left-hand side depends only on r, while the right-hand side depends only on θ and ϕ. Both sides must therefore be equal to the same constant. Writing that constant as $-\ell(\ell+1)$, we obtain the angular equation

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2Y}{\partial\phi^2}=-\ell(\ell+1)\,Y(\theta,\phi), \tag{13}$$

and the radial equation

$$-\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{2mr^2}{\hbar^2}\left[V(r)-E\right]=\ell(\ell+1). \tag{14}$$

Equation (Eq. 13) is the familiar eigenvalue problem for orbital angular momentum,

$$\widehat{L}^2Y_\ell^m(\theta,\phi)=\ell(\ell+1)\hbar^2\,Y_\ell^m(\theta,\phi), \tag{15}$$

whose solutions are the spherical harmonics $Y_\ell^m(\theta,\phi)$.

It is convenient to define the reduced radial function $u(r) = rR(r)$. The radial equation then becomes

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2mr^2} + V(r)\right]u(r) = E\,u(r), \tag{16}$$

which is exactly the standard one-dimensional radial Schrödinger equation in an effective potential. The important point is that nothing in the spatial derivation changes. The energy-domain formulation leaves the angular and radial sectors untouched; all of the novelty is carried only by the global phase in (Eq. 2).

3 Observables in the Energy-Domain Picture

Once the wavefunction has been written in the form of (Eq. 2), the interpretation of observables becomes straightforward. The whole observable content is encoded in the spatial eigenfunction $\psi(\mathbf{r})$, while the parameter z appears only through a phase factor.

We write the full state as

$$\Psi(\mathbf{r}, E) = \psi(\mathbf{r})\,\exp\left(-\frac{iE^2z}{\hbar}\right), \tag{17}$$

with $\psi(\mathbf{r})$ solving the stationary Schrödinger equation. The corresponding probability density is therefore

$$\rho(\mathbf{r}) = |\Psi(\mathbf{r}, E)|^2 = |\psi(\mathbf{r})|^2, \tag{18}$$

which is independent of z. This means that the spatial distribution of the particle is exactly the same as in the usual stationary formulation.

The same conclusion holds for expectation values. If $\widehat{A}$ is any operator acting on the spatial degrees of freedom, then

$$\langle A\rangle = \int \psi^*(\mathbf{r})\,\widehat{A}\,\psi(\mathbf{r})\,d^3\mathbf{r}, \tag{19}$$

so position, momentum, angular momentum, and energy expectation values are unchanged. The extra phase cancels because it is global.

The generator that appears in the present construction is

$$\widehat{H}_b = \frac{i\hbar}{2z}\frac{\partial}{\partial E}, \tag{20}$$

and its role is only to reproduce the energy eigenvalue through the derivative of the quadratic phase. It does not alter the spatial sector of the theory. In that sense it plays a role similar to the factor $\exp(-iEt/\hbar)$ in ordinary stationary quantum mechanics: it carries the evolution label, but it does not change the measurable content of an energy eigenstate.

The operator algebra also remains the same. In particular, the canonical commutation relations

$$[\hat{x}_i, \hat{p}_j] = i\hbar\,\delta_{ij}, \tag{21}$$

and the usual angular momentum commutation relations are not modified. The present formulation is therefore not a replacement for the kinematics of quantum mechanics. It is a different way of organizing the same kinematics and the same stationary dynamics.

For that reason, the auxiliary parameter z should not be interpreted as a new physical coordinate. In this paper it is only an evolution label. Changing z changes the overall phase of the state, but it does not move the particle, change the spectrum, or alter any expectation value. Different values of z therefore correspond to the same physical stationary state written with a different global phase convention.

The physical meaning of the construction is then clear. The energy-domain formulation is not introducing new observables. Instead, it makes explicit that for stationary systems the measurable physics already sits entirely in the spatial eigenvalue problem. The role usually assigned to temporal phase evolution can be rewritten as an energy-domain phase evolution without changing what can actually be observed.

4 Comparison with the Standard Time-Dependent Formulation

It is useful to compare the present construction directly with the standard time-dependent Schrödinger equation. This makes the relation between the

two formulations transparent and shows more clearly what has changed and what has not.

In ordinary quantum mechanics, the state evolves according to

$$i\hbar\frac{\partial\Psi(\mathbf{r},t)}{\partial t} = \widehat{H}\Psi(\mathbf{r},t), \tag{22}$$

where $\widehat{H}$ is the Hamiltonian. If the Hamiltonian is time-independent, the solution can be separated as

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r})\,\exp\left(-\frac{iEt}{\hbar}\right), \tag{23}$$

with $\psi(\mathbf{r})$ satisfying

$$\widehat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}). \tag{24}$$

For stationary states, all measurable information is contained in $\psi(\mathbf{r})$, and the factor $\exp(-iEt/\hbar)$ contributes only a phase.

The same pattern appears in the energy-domain picture. There we write

$$\Psi(\mathbf{r},E) = \psi(\mathbf{r})\,\exp\left(-\frac{iE^2z}{\hbar}\right), \tag{25}$$

and the spatial function $\psi(\mathbf{r})$ again solves the stationary eigenvalue equation. The structure is therefore parallel: in both descriptions the state factorizes into a spatial eigenfunction and a phase factor, and in both descriptions the observable content is carried by the spatial part.

The difference lies in the parameter used to write that phase. In the standard theory the phase is written in terms of the external time parameter t. In the present approach the phase is written in terms of the energy variable E and the auxiliary label z. The parameter z is not interpreted as an observable coordinate; it simply indexes the phase in the energy representation.

From the formal point of view, the evolution law

$$\frac{i\hbar}{2z}\frac{\partial\Psi}{\partial E} = \widehat{H}\Psi, \tag{26}$$

plays the role analogous to the time derivative in (Eq. 22). Because the derivative acts only on the exponential factor in (Eq. 25), the spatial eigenfunctions are left unchanged. This is the reason the two formulations agree on spectra, expectation values, and all stationary observables.

The correspondence is especially clear for a bound system such as hydrogen. The geometry of the orbitals, the angular momentum quantum numbers, the degeneracies, and the radial probability densities all come from the spatial equation. Whether one writes the global phase with t or with the pair (E, z) does not alter any of that structure.

The conclusion of the comparison is therefore straightforward. For time-independent Hamiltonians, the present formalism should be understood as an equivalent representation of standard quantum mechanics, not as a competing dynamical theory. It removes explicit time from the formulation, but it does not remove or change the physical predictions of the stationary theory.

5 Hydrogen Atom in the Energy-Domain Picture

We now apply the formalism to the hydrogen atom. This is the most important test in the paper, because hydrogen combines a central Coulomb potential, nontrivial angular momentum, and a discrete spectrum known exactly from standard quantum mechanics.

The interaction between the electron and the proton is described by the Coulomb potential

$$V(r) = -\frac{e^2}{4\pi\varepsilon_0 r}, \tag{27}$$

which depends only on the radial coordinate. Substituting this into the general radial equation (Eq. 16) gives

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[\frac{\hbar^2\ell(\ell+1)}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}\right] u(r) = E\,u(r), \tag{28}$$

where $u(r) = rR(r)$ is the reduced radial function.

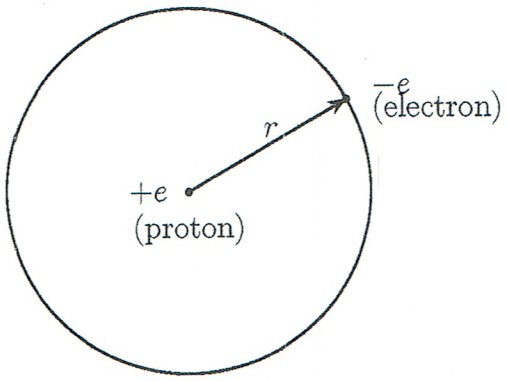

Figure 2: The hydrogen atom.

This equation has exactly the same form as in the standard stationary treatment [3, 8]. The solution therefore follows the familiar sequence of steps, which we summarize here because it shows clearly that no reference to time is needed anywhere in the derivation.

For bound states we have $E < 0$. We define

$$\kappa = \sqrt{\frac{-2mE}{\hbar^2}}, \tag{29}$$

and first study the limiting behaviour of the radial equation. As $r \to \infty$, the Coulomb term becomes negligible and we obtain

$$\frac{d^2u}{dr^2} \approx \kappa^2 u, \tag{30}$$

whose normalizable solution behaves as

$$u(r) \sim e^{-\kappa r}. \tag{31}$$

Near the origin, the centrifugal term is dominant. Keeping only the leading contribution gives

$$\frac{d^2u}{dr^2} \approx \frac{\ell(\ell+1)}{r^2} u, \tag{32}$$

so the regular solution behaves as

$$u(r) \sim r^{\ell+1}. \tag{33}$$

These two limits suggest the standard ansatz

$$u(r) = r^{\ell+1} e^{-\kappa r} v(r), \tag{34}$$

where $v(r)$ is expanded in a power series. Introducing the dimensionless variable

$$\rho = 2\kappa r, \tag{35}$$

and substituting the ansatz into (Eq. 28) leads to the associated Laguerre equation. Normalizability requires the series to terminate, and that condition produces the quantization of the bound states.

Writing the radial quantum number as $n_r = 0, 1, 2, \ldots$, the principal quantum number is

$$n = \ell + 1 + n_r, \qquad n = 1, 2, 3, \ldots, \tag{36}$$

and the termination condition may be written as

$$\kappa = \frac{1}{n a_0}, \tag{37}$$

where the Bohr radius is

$$a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m e^2}. \tag{38}$$

Substituting (Eq. 37) into (Eq. 29) gives the hydrogen spectrum

$$E_n = -\frac{m e^4}{2(4\pi\varepsilon_0)^2 \hbar^2} \frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots, \tag{39}$$

which is the standard textbook result.

The normalized radial wavefunctions are

$$R_{n\ell}(r) = \mathcal{N}_{n\ell} \left(\frac{2r}{n a_0} \right)^{\ell} e^{-r/(n a_0)} L^{2\ell+1}_{n-\ell-1} \left(\frac{2r}{n a_0} \right), \tag{40}$$

where $\mathcal{N}_{n\ell}$ is the normalization constant. Combining the radial and angular parts gives

$$\psi_{n\ell m}(\mathbf{r}) = R_{n\ell}(r)\, Y_\ell^m(\theta, \phi), \tag{41}$$

with the usual degeneracy in ℓ and m for fixed n.

Finally, the full wavefunction in the energy-domain formulation is obtained by restoring the phase factor,

$$\Psi_{n\ell m}(\mathbf{r}, E) = \psi_{n\ell m}(\mathbf{r})\, \exp\left(-\frac{iE^2 z}{\hbar}\right). \tag{42}$$

This factor depends only on the energy eigenvalue and the auxiliary parameter z. It does not modify probability densities, selection of bound states, angular momentum structure, or expectation values.

The conclusion of the hydrogen analysis is therefore direct. The energy-domain formulation reproduces the full hydrogen problem exactly: the same radial equation, the same eigenfunctions, the same degeneracies, and the same discrete energies. Replacing time by the auxiliary parameter z changes only the global phase convention.

6 Scope and Possible Extensions

The hydrogen atom shows clearly what the present formulation can do. It reproduces a complete three-dimensional bound-state problem without introducing time as a fundamental variable. It is therefore useful to state carefully the range over which this conclusion holds.

The essential structural point is the factorization

$$\Psi(\mathbf{r}, E) = \psi(\mathbf{r})\, \exp\left(-\frac{iE^2 z}{\hbar}\right), \tag{43}$$

together with the stationary eigenvalue problem

$$\widehat{H}\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{44}$$

from which the measurable content of the theory is extracted. Whenever a system is described by a time-independent Hamiltonian and admits such a

stationary decomposition, the present energy-domain representation reproduces the same spectrum and the same eigenstates as the standard formulation.

This means that the construction applies naturally to a broad class of stationary bound-state systems, including other central potentials and multidimensional separable problems. In all such cases, the quantization comes from boundary conditions, symmetry, and normalizability. It does not require time to appear explicitly in the formalism.

At the same time, the present paper is intentionally limited to time-independent Hamiltonians. That is not a weakness of the derivation; it is part of the point. For stationary systems, the usual time dependence already reduces to a phase factor. The energy-domain formulation makes that fact explicit by moving the phase structure into the energy representation.

To go beyond this class of problems, more work is needed. Explicitly time-dependent Hamiltonians would require an energy-domain analogue of ordered evolution. Scattering states would require a treatment of continuum normalization, asymptotic states, and transition amplitudes in energy space. These questions are important, but they lie beyond the scope of the present paper.

Conceptually, the result is still significant. It shows that for a physically important class of systems, time is not needed to recover the measurable predictions of nonrelativistic quantum mechanics. The observables come from the stationary spatial sector, while the auxiliary parameter z keeps track only of an unobservable phase.

7 Hydrogen Spectrum

After deriving the hydrogen eigenstates, it is useful to display the spectrum explicitly. This gives a direct check that the energy-domain formalism recovers not only the abstract eigenvalue problem, but also the familiar numerical structure of the atom.

From (Eq. 37) and (Eq. 39), the allowed energies are

$$E_n = -\frac{me^4}{2(4\pi\varepsilon_0)^2\hbar^2}\frac{1}{n^2}, \qquad n = 1, 2, 3, \ldots. \tag{45}$$

This is the Bohr spectrum obtained from the full Schrödinger solution.

Using the accepted constants, the first few levels are

$$E_1 = -13.6 \text{ eV}, \qquad E_2 = -3.40 \text{ eV}, \qquad E_3 = -1.51 \text{ eV}. \tag{46}$$

More generally, the levels follow the familiar $E_n \propto -1/n^2$ law and accumulate at the ionization limit $E = 0$.

7.1 Spectral lines from energy differences

When the atom makes a transition from an initial state n_i to a lower final state n_f, the emitted photon energy is

$$E_\gamma = E_i - E_f. \tag{47}$$

Substituting the hydrogen energies gives

$$E_\gamma = 13.6 \text{ eV} \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \qquad n_i > n_f. \tag{48}$$

Relating photon energy to frequency and wavelength through

$$E_\gamma = h\nu, \qquad \nu = \frac{c}{\lambda}, \tag{49}$$

one obtains the standard Rydberg formula

$$\frac{1}{\lambda} = R \left(\frac{1}{n_f^2} - \frac{1}{n_i^2} \right), \tag{50}$$

with

$$R = \frac{me^4}{8\varepsilon_0^2 h^3 c}. \tag{51}$$

The important point is that all of these results follow from the same discrete spatial eigenvalues obtained earlier. The auxiliary parameter z enters only through the global factor in (Eq. 2), so it cannot change any spectral difference.

7.2 Energy-level diagram and series

Figure 3 shows the first few levels together with representative transitions in the Lyman ($n_f = 1$), Balmer ($n_f = 2$), and Paschen ($n_f = 3$) series. The figure is drawn directly from the energies in (Eq. 45) and therefore provides a visual summary of the standard spectroscopic structure recovered by the present approach.

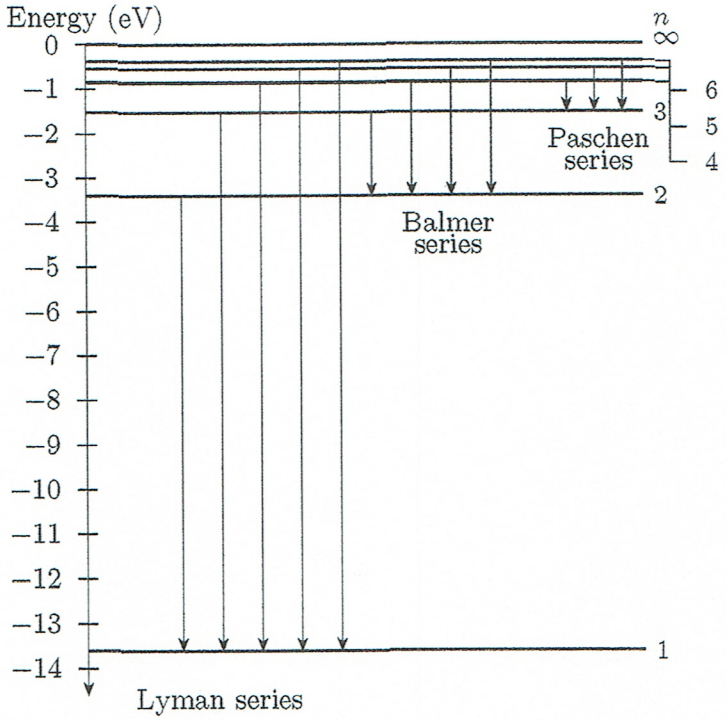

Figure 3: Energy levels and representative transitions in the spectrum of hydrogen.

The experimentally known hydrogen spectrum is therefore recovered in full. The absolute level energies, the transition energies, and the standard spectral series all follow from the same stationary eigenvalue problem as in ordinary quantum mechanics.

8 Conclusions

In this paper we extended the time-free formulation introduced in our earlier work to three-dimensional nonrelativistic quantum mechanics and applied it to the hydrogen atom. The central result is that the energy-domain evolution law separates into a stationary spatial eigenfunction and a global phase depending on E and z. Because of that separation, the spatial part of the theory remains identical to the usual stationary Schrödinger problem.

We then showed that the standard derivation in spherical coordinates goes through without modification. The angular equation still gives the spherical harmonics $Y_\ell^m(\theta, \phi)$, the reduced radial equation keeps its familiar effective-potential form, and for the Coulomb potential one recovers the full hydrogenic spectrum and radial eigenfunctions. The principal quantum number, the ℓ and m degeneracies, and the associated Laguerre polynomial solutions are all the same as in the conventional theory.

We also examined the role of observables and found that the parameter z has no direct measurable meaning in the present framework. It contributes only a global phase, so probability densities, expectation values, and spectral differences are unchanged. In this sense, the formulation is not a modification of quantum mechanics for stationary systems. It is an equivalent representation in which explicit time has been removed.

The hydrogen spectrum provides the clearest check of this statement. Once the Coulomb eigenvalue problem is solved, the Bohr energies, the transition energies, the Rydberg formula, and the Lyman, Balmer, and Paschen series all follow in the usual way. The spectrum therefore confirms at the observable level what the formal analysis already suggested.

The natural next steps are also clear. A full treatment of explicitly time-dependent Hamiltonians, scattering states, and transition amplitudes would require additional development in the energy-domain picture. Even so, the present result already shows that for a broad and important class of time-independent systems, quantum mechanics can be written consistently without taking time as a fundamental parameter.

Declarations

The authors declare that they have no competing interests and that no external financial support was required for this work.

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